Gödel and the gap in mathematics

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1 Mathematics and meta-mathematics

1.1 Formalising mathematics

At the close of the nineteenth century, an exciting development was going on in mathematics. The brilliant German mathematician David Hilbert published his '*Grundlagen der Geometrie*' in 1899, a book which for the first time formalised a large part of mathematics – geometry – using the modern axiomatic method.[1] What does this method consist of? One specifies, in a certain technical language, a finite number of axioms or axiom-schemata; one furthermore provides rules of reasoning, whereby new propositions can be formed from older ones; and thus a completely unambiguous system of reasoning is created. Proofs consist of manipulating strings of symbols according to the specified rules, until the desired string is reached – the validity of every step of reasoning, being formalised, can be verified by simple inspection. Truths deduced by formal mathematical reasoning have the highest certainty possible.

Formalising a part of mathematics often creates an astounding amount of clarity where much confusion existed before. Ernest Nagel and James R. Newman sing the praise of this method in their 1959 book 'Gödel's Proof':

> Formalization is a difficult and tricky business, but it serves a valuable purpose. It reveals structure and function in a naked clarity, as does a cut-away working model of a machine. When a system has been formalized the logical relations between mathematical propositions are exposed to view; one is able to see the structural patterns of various "strings" of "meaningless" signs, how they hang together, how they are combined, how they nest into one another, and so on.[2]

The price to be paid for this achievement, however, is a ruthless abstraction from all meaning. In the formal system nothing is left but the structure of the provable formulae; there is no hidden residue of meaning, except perhaps in the mathematicians mind. The formal system works by virtue of the existence of an *interpretation* of its formulae; whereas meaning is irrelevant to proving theorems, the theorems thus proved can be *assigned* a meaning and thus converted into mathematical statements. A formalisation of a part mathematics is complete if and only if *all* truths and *only* truths of the mathematical subject are represented by provable formulae of the axiomatic system. Yet for a formal mathematical system to be clear and unambiguous in the sense described above, as well as complete, one condition has to be fulfilled: the mathematical subject in question must be isolable from all other meaningful discourse. The formal system is isolated from all external influences; thus if it completely describes a part of mathematics, an understanding of this subject must be possible apart from all other subjects.

[1] [1], p. 54.
[2] [3], p. 27.

1.2 The gap in mathematics

A formal system contains, or consists of, strings of symbols; and it is evidently the case that we can talk about such strings. We can claim that one is longer than the other, that they contain these and these symbols, etcetera. Some strings are contained in the formal system, others are not; perhaps we find out that all strings are accompanied by a string which is identical except for one extra symbol in front of it. All of this discourse can take place prior to the interpretation of the strings as mathematical statements. Hence, once we do assign meaning to the strings, we transform two systems of propositions at once: the propositions *of* the formal system are transformed into mathematics, and the propositions *about* the formal system are transformed into meta-mathematics. According to Nagel and Newman:

> Meta-mathematical statements are statements about the signs occurring within a formalized mathematical system (i.e., a calculus) – about the kinds and arrangements of such signs when they are combined to form longer strings of marks called "formulas," or about relations between formulas that may obtain as a consequence of the rules of manipulation specified for them.[3]

Looking at arithmetic, the part of mathematics concerned with adding and multiplying natural numbers, the following are (true) mathematical statements:

$$7 + 11 = 18$$
$$3 * 8 = 24$$
$$\forall x : 0 \neq x + 1$$

Examples of meta-mathematical statements in arithmetic are:

'$7 + 11 = 18$' is a provable formula.
'$\forall x : 0 \neq x + 1$' does not contain free variables.
Arithmetic is consistent.

There must be a fundamental difference between mathematical and meta-mathematical statements in a certain subject if an axiomatic formal system completely represents it. The distinction between them is analogous to that between the formal system and discourse about it; it is that between a subject matter (for example, arithmetic) and the discourse about the subject matter. A formal system is, as we have seen, completely isolated from the discourse about it; therefore, a mathematical subject can only be represented by such a system if it is isolable from meta-mathematics.[4]

1.3 Is mathematics split?

Does this distinction between mathematics and meta-mathematics accurately reflect a property of the activity of mathematicians? It it useful and important, or is it a misleading artefact of formalisation? Nagel and Newman take the first view, when they write:

[3] [3], p. 28.

[4] Note that the terminology adopted here is not universally accepted. Someone like Curry, who claims that mathematics is the science of formal systems ([1], p. 53.), will find the present use of the word 'mathematics' misleading. For him, mathematics is what I have called meta-mathematics; and what I call mathematics he'll call – more or less – a formal system. The 'more or less' clause is necessary because within the present context it is meaningful to ask whether a certain formal system completely represents a certain part of mathematics; but in Curry's language such a question cannot meaningfully be asked – at least not without extensive re-interpretations. Formalists like Curry will find the present paper misguided, since it assumes that there is a subject apart from the formal system which can be represented more or less successfully.

> The importance to our subject of recognizing the distinction between mathematics and meta-mathematics cannot be overemphasized. Failure to respect it has produced paradoxes and confusion. Recognition of its significance has made it possible to exhibit in a clear light the logical structure of mathematical reasoning. The merit of the distinction is that it entails a careful codification of the various signs that go into the making of a formal calculus, free of concealed assumptions and irrelevant associations of meaning.[5]

On their view, then, mathematics acquires its most clear and pure form when codified into a formal language. There is a radical distinction between mathematical reasoning – which is codified in the formal system – on the one hand, and reasoning about mathematics (and reasoning about mathematical reasoning) on the other. But if this distinction is so important, why wasn't it recognised before the early twentieth century? Is it really true that we can find two autonomous discourses in the mathematician's language, one of which consists of formal propositions while the other speaks about those propositions? The recency of the proposed distinction makes this initially implausible; the success of the axiomatic method in mathematics could, however, decide the case in favour of Nagel's and Newman's position. I wish to suggest in the rest of this paper that the most spectacular failure of the axiomatic method – that shown in the incompleteness theorems of Gödel – can in fact be understood as a strong argument *against* the usefulness of the 'mathematics/meta-mathematics'-dichotomy.

2 Gödel's famous paper

In the heyday of the formal axiomatic approach, a young mathematician named Kurt Gödel published an article with the rather unattractive title '*Über formal unentschiedbare Sätze der Principia Mathematica und verwandter Systeme I*'[6]. Its content more than made up for its uninventive name, as the paper became a much-read and more-discussed classic in mathematics and logic. Gödel's results are often summarised as follows: "Every formal axiomatic system which is strong enough to represent arithmetic is, if consistent, incomplete. (A system is consistent if not every well-formed formula is provable; and complete if every true formula is provable.) And the consistency of any such system is, if it actually is consistent, unprovable in the system itself." A more detailed version of the argument for the first of these claims will be given presently; for the fully detailed technical account, the reader is referred to [2].

Assume that arithmetic has been formalised in a formal system, P. We now have two discourses: the discourse of arithmetic, which consists of the formulae of P; and the discourse about arithmetic (and P), for which a subset of a natural language is used. Gödel's aim is to represent part of the latter discourse in the former, in such a way that if a formula in P, say X_P, represents a proposition in the natural language, say $X_{natural}$, X_P is true if and only if $X_{natural}$ is true. In this way the represented meta-mathematical reasoning would be exactly copied in mathematical reasoning.

To achieve this aim, Gödel associates a natural number with every proposition in P; thus, to every formula X in P there belongs a number which is called the Gödel-number of X, $G(X)$. Arithmetic contains propositions about natural numbers, so through this act of representation the possibility is created for P to speak

[5] [3], p. 28.

[6] This essay is based on the English translation [2].

about its own formulae. Take, for instance, the proposition 'The formula X contains fewer than 10 symbols.'. There is a finite class of formulae for which this is true, and each of these has a Gödel-number associated with it. If we could find a class of natural numbers C which contained those and only those numbers which are the Gödel-numbers of formulae with fewer than 10 symbols, we would have two equivalent statements: 'The formula X contains fewer than 10 symbols.' would be equivalent to '$G(X) \in C$'. The latter is a purely arithmetical statement, the former is a meta-mathematical one. Gödel goes on to show that for many meta-mathematical statements, such corresponding mathematical statements can indeed be constructed. Among the meta-mathematical concepts he successfully represents in P are 'variable', 'negation', 'formula', 'substitution', 'axiom' and 'proof'. All meta-mathematical statements containing only formulae of P and these concepts can be represented in P itself; and their truth or falsity is conserved by this representation. But things get really going when Gödel finally defines the notion of 'provability' in P.

This allows him to construct a proposition $NP(x)$ in P, which has the meta-mathematical meaning 'The formula with Gödel-number x is not provable.'. Thus, for any formula we can insert its Gödel-number into this proposition, and obtain a proposition in P which represents the statement that the formula of our choice is not provable. This is the first important ingredient.

Every symbol has a Gödel-number, by virtue of the fact that every string of symbols whatever has a Gödel-number. Let the symbol y be a part of P, which acts as a variable and has the Gödel-number 13. Now we define $sub(y, 13, y)$ as the Gödel-number of the formula which is obtained when in the formula with Gödel-number y, every symbol with Gödel-number 13 is changed into the symbols for the number y. In other words, take the formula with Gödel-number y; replace every 'y' in this formula with the number y (the *symbol* 'y' is changed to the symbols for the *number* y); and calculate the Gödel-number of this new formula. This is the second ingredient we need.

We now define the formula N as follows: $N \equiv NP(sub(y, 13, y))$. In words, N is the proposition that says that the formula with Gödel-number $sub(y, 13, y)$ is not provable. Or, equivalently, that the formula which is obtained when in the formula with Gödel-number y, every symbol with Gödel-number 13 is changed into the symbols for the number y, is not provable. Since N is a formula of P, it has a Gödel-number; we'll call it n. Now we can formulate the famous 'Gödel-sentence':

$$G \equiv NP(sub(n, 13, n)) \tag{1}$$

In words, G says: the proposition that you get when you substitute 'n' for 'y' in the sentence with Gödel-number n is not provable. We know that the formula with Gödel-number n is N, so G amounts to: the formula that you get when you substitute 'n' for 'y' in N is not provable. Yet when you substitute 'n' for 'y' in N, you get G, as one look at N and G will learn us. So in the most simplified form G says: G is not provable.

If this formula is provable, it is false; but all provable formula must be true, so this is impossible. Therefore it cannot be provable, which ensures that it is true. G is a true proposition in P which cannot be proven, if P is consistent. There is a statement in arithmetic that cannot be proven in the formal system P – hence, P is incomplete. This conclusion holds for a huge class of formal systems; in particular it holds for all stronger systems which are created by adding more axioms to P. There will always be true arithmetical statements that are not provable in the formal system.

3 Closing the gap

How could this surprising analysis be interpreted as an argument against the importance of the distinction between mathematics and meta-mathematics?[7] Gödel's ingenious strategy for proving the incompleteness of P starts with creating a strong link between mathematical and meta-mathematical statements. This may cast doubt upon the difference between the two discourses, but is not yet a counter-argument to the 'theory of the gap' – Nagel's and Newman's conviction that mathematics and meta-mathematics can and should be separated.[8] But he then succeeds in identifying a proposition G in arithmetic which cannot be proven in the formal system, but which is nevertheless *true*. It is important to note that the truth or falsity of G cannot be ascertained within P. The system simply says nothing at all about this proposition. But we know that G is a true statement because of *meta-mathematical considerations*; it is shown to be true in virtue of our interpretation of G as a meta-mathematical statement on provability. We thus have found a mathematical statement the truth of which can only be ascertained by meta-mathematical considerations.

But this means that the gap has been bridged; or rather, this shows that there was no gap in the first place. When we try to isolate a part of the mathematician's discourse as 'mathematics' – in the way that Nagel and Newman urge us to do – we will fail; the remaining part of the mathematician's language still has something important to say on the topic we've tried to isolate. Some mathematical truths are not present in 'mathematics', but are present in the combination of 'mathematics' and 'meta-mathematics'; no matter how we place the gap, some mathematical truths are not on the side we labeled 'mathematics'. The picture of a subject and a discourse about that subject is simply flawed; the so-called 'discourse' will always contain a part of the subject under consideration.[9] Perhaps it is indeed the case that '[f]ailure to respect [the gap] has produced paradoxes and confusion'. But failing to respect the distinction between the mathematical and the meta-mathematical is exactly what Gödel has done in his famous and acclaimed paper – and one may wonder how much respect we owe to a flawed artefact anyway.

References

[1] **Dalen, D. van**, *Filosofische grondslagen van de wiskunde*, Van Gorcum & Comp., 1978

[2] **Gödel, K.**, *On Formally Undecidable Propositions of Principia Mathematica And related Systems*, translated by B. Meltzer, with an introduction by R. B. Braithwaite, Oliver & Boyd, 1962

[3] **Nagel, E. & James R. Newman**, *Gödel's Proof*, Routledge & Kegan Paul Ltd., 1959

[7] It should be stressed that I do not assert that this is its *only* content, or its only philosophical significance. But I think it is one of the insights offered by Gödel's paper.

[8] Or, put perhaps more clearly, that 'mathematics' and 'meta-mathematics' are meaningful terms without overlap.

[9] All this applies only to formal systems strong enough to represent arithmetic, of course. For many weaker systems, a gap can be found. These are, however, often mathematically uninteresting.