OF ABSTRACT BEAUTY
Mathematics and Kant's 'free play'

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1 Introduction

> The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree.
> – Aristotle, *'Metaphysics', Book XIII, Chapter 3*

Most modern mathematicians will unhesitatingly agree with Aristotle's claim that mathematics contains much beauty. The famous mathematician Henri Poincaré even claimed that the aesthetic, rather than the logical, is the dominant element in mathematical creativity.[1] Yet there seems to be a great difference between mathematical theorems and theories on the one hand, and the objects to which we normally assume aesthetic judgements to apply on the other: paintings, sculptures, movies, music, books, poems, and so forth. This might raise interesting questions as to the precise nature of the aesthetic component of mathematics, but it may also casts doubt upon the idea that there is such a component in the first place. If the mathematical and the artistic are completely unalike, why should we describe them using the same concepts? Is it obvious that the word 'beautiful' can apply not only to Mozart's *Don Giovanni*, but also to Gödel's incompleteness theorems?

Such considerations may lead to attempts to show that mathematicians who describe their creations as 'beautiful' are in fact using the wrong word; perhaps 'elegant', 'efficient' or 'analysable' would do a better job. To counterbalance the overwhelming support from within the mathematical community for the idea that 'beautiful' *is* the right term to use, this project would have to provide us with a tightly structured argument analysing the concept of 'beauty' and showing that all these people are wrong. For such an analysis to be convincing, it has to be grounded in a well-developed theory of aesthetics. Therefore, a natural strategy is to look to the philosophy of Kant, who created perhaps the most important of such theories. It will be my object in this essay to construct an argument against the beauty of mathematics on the basis of the concept of the 'free play of the cognitive powers' in Kant's *Kritik der Urteilskraft*; and then to show its failure. This should not be interpreted as a fight against windmills of my own making, but rather as an attempt to clear away a deluding intuition that may tempt people to dismiss the aesthetic component of mathematics as a fiction.

[1] [2], p. 168.

2 The Kantian argument

2.1 The free play of cognitive powers

Kant's *Kritik der Urteilskraft* presents a fully developed theory of aesthetics and art. One of the subjects that concerns Kant is the nature of the aesthetic judgement '*x is beautiful*'. Is it subjective or objective? Universal or particular? A priori or a posteriori? Central to his analysis of aesthetic judgements is the twofold claim that on the one hand there is no rule with which we can deduce from the objective properties of an object whether or not it is beautiful, whereas on the other hand we *do* feel that aesthetic judgements are universal in the sense that everyone should agree on them. That the aesthetic judgement cannot be deduced from the objective properties of the object is very easily seen: when asked to judge whether some object is beautiful, we'll never be content with just a description of it, no matter how detailed. Before we judge, we want to see it with our own eyes (or use whatever other sensory apparatus is needed to perceive the object). When we are told of an 80-feet chair, we'll have no qualms in saying that it's *huge*; when someone describes the contents of a book to us, we might well agree with him that it's *interesting*; but no mere list of properties can convince us that anything is *beautiful*. And yet we think that judgements of beauty are universal in a sense that those of appreciation are not. If I like tea, but you hate it, we will not quarrel about it. There is no tension between my appreciation and your dislike – it is 'just a matter of taste'. But when I claim that something is beautiful, I claim more than just that I experience some purely subjective feeling. I implicitly claim that every discerning man and woman should agree with me, that anyone with a right to judge should judge in the same way that I do. The aesthetic judgement contains a demand that everyone agree with us.

Thus the judgement of beauty cannot be reduced to an analysis of concepts, but it must be in some way universal and communicable to other humans – but only the cognitive is communicable, and cognition is the realm of concepts. There is an obvious tension between the non-conceptual and the conceptual part of beauty's nature. This tension can be neutralised, Kant argues, by recognising that since the aesthetic judgement must have to do with cognition but cannot reduce to any cognitive content, it must be based on *cognition as such*. The positive sensation of beauty we feel when looking at an object is the pleasure which arises not from contemplating any concepts, but from the *free play of the cognitive powers*. Anything is beautiful if and only if it arouses in us such a free play, an activity of cognition unrestricted by any definite concepts. Kant himself writes (paragraph 9 of the *Kritik der Urteilskraft*):

> Die Erkenntniskräfte, die durch diese Vorstellung ins Spiel gesetzt werden, sind hierbei in einem freien Spiele, weil kein bestimmter Begriff sie auf eine besondere Erkenntnisregel einschränkt.

The representation which is the subject of such a free play cannot be a definite object of knowledge, as that would destroy the free play. The moment our thoughts determine the object in strict concepts, there can no longer be any unrestricted activity; only if we can engage in a free play the object deserves the name 'beautiful'. Thus, any representation which is determinate cannot be the basis of a positive aesthetic judgement. Kant formulates it thus (again paragraph 9 of the *Kritik der Urteilskraft*):

> Wäre die gegebene Vorstellung, welche das Geschmacksurteil veranlaßt, ein Begriff, welcher Verstand und Einbildungskraft in der Beurteilung des Gegenstandes zu einem Erkenntnisse des Objekts vereinigte, so wäre das Bewußtsein dieses Verhältnisses intellektuell (wie im objektiven Schematism der Urteilskraft, wovon die Kritik handelt). Aber das Urteil wäre auch alsdann nicht in Beziehung auf Lust und Unlust gefällt, mithin kein Geschmacksurteil.

2.2 Mathematics versus the free play

This short recapitulation of a part of Kant's theory has supplied us with excellent tools to create an argument purporting to show that mathematics has no aesthetic content. No representation can be beautiful if it is a fully determinate object of knowledge, since such an object allows the mind no freedom. There cannot be a free play of cognitive powers, because the representation is already determinate; the mind can think about it in one way and one way only, there are no alternatives. And what, we might continue, is more determinate than mathematics? Most of its theories can be written down in axiomatic form, where all theorems follow from the axioms by the strict laws of logical analysis. Every truth, all meaning is laid down from the outset, in a complete non-ambiguous form, in the axioms and the logical laws; a few strings of symbols determine every aspect of the mathematical theory.[2] What could possibly be a *less* likely candidate for an object that could initiate a free play of the cognitive powers?

But without a free play, there can be no beauty. Whatever mathematics is, it is not beautiful – simply because it does not allow in even the slightest way for the free play of cognitive powers. The mathematical community has been wrong all along, and should think up a new word to describe their emotional bond with their work – we already proposed 'elegant', 'efficient' and 'analysable'. Or should they?

3 Mind, logic and determinateness

3.1 Mathematical beauty

Let's take a few steps backwards, and ask the mathematician to tell us about what, according to him, is, and what is not beautiful in mathematics. There is, after all, the slight possibility that we have overlooked some subtlety that undercuts our argument. The mathematician tells us that an excellent example of something beautiful is the theory known as Peano arithmetic, which is that part of mathematics that describes the natural numbers, i.e. the numbers $0, 1, 2, 3, \ldots$ A great feature of this theory, we are told, is that it is expressible in only five axioms, which in informal notation look thus:

[2] This characterisation of mathematics is a gross oversimplification. More importantly, it is very much at odds with the views of influential philosophers of mathematics as Edmund Husserl and L.E.J. Brouwer. However, an exposition of the differences between formalism and intuitionism would take up too much space, and would not make the present discussion any clearer. I will therefore continue as if the characterisation is unproblematic.

1. There is a natural number 0.
2. Every natural number a has a successor, denoted by $a + 1$.
3. There is no natural number whose successor is 0.
4. Distinct natural numbers have distinct successors: if $a \neq b$, then $a + 1 \neq b + 1$.
5. If a property is possessed by 0 and also by the successor of every natural number it is possessed by, then it is possessed by all natural numbers.

We also want to add and multiply natural numbers, which requires an additional four axioms:

6. $a + 0 = a$
7. $a + (b + 1) = (a + b) + 1$
8. $a * 0 = 0$
9. $a * (b + 1) = (a * b) + a$

With the aid of this system, we can deduce well-known truths such as *2 + 2 =4* and *the sum of two odd numbers is even.* These propositions are completely determinate, and do not allow for a free game of our cognitive powers. We tell the mathematician as much. 'Ah!', he'll say, 'you are quite right to say that those propositions are not beautiful. It is *the whole of the theory* which is beautiful, with its vast amount of theorems, its unknown corners, its many riddles and exciting puzzles, its surprising connections, and so forth! Any theorem is, well, just a theorem – in and of itself it almost never contains any beauty. But the structure of the whole is an object of the deepest possible aesthetic contemplation.'

Important as it may be to understand that beauty is not claimed for individual propositions like *2 + 2 =4* but for Peano arithmetic as such, our argument seems untouched. The theory as a whole is just as determinate as the individual theorems, so no free play is possible. Peano arithmetic cannot possibly be beautiful. However, we should take a good look at the meaning of 'determinate'.

3.2 Logical and mental determinateness

The concepts of Peano arithmetic are completely determinate in the sense that their meaning and all the ways in which they can appear in propositions are determined by the small set of axioms above, plus the rules of logic. There is no ambiguity of meaning, and whether anything is true or false is crystal clear. At least, it is in a *logical* sense. The mathematical theory seen as a structure in logical space is clear and well-determined. But the theory as a *mental* object in the mind of the mathematician is rather less clear and determinate. First of all, whereas it is logically clear for any proposition expressible in the theory's language whether it is true, false or perhaps neither, this clarity is obviously not mirrored in the mathematician's mind. It is regularly unclear whether a given proposition is true or false, and discovering this is one the mathematician's main concerns. A large part of doing mathematics is finding proofs or disproofs of theorems, which is as often as not an exploration with the aim of discovering *whether* something is or is not true. This is the case even within a theory as simple as Peano-arithmetic. How is this possible? All the basic concepts are logically clear, and presumably even mentally clear: we know what natural numbers are, what addition is and what multiplication is. Since the truth or falsity of a mathematical statement is generally thought to be analytic a priori, it is contained within the concepts employed and the structure of the proposition; when the concepts are clear, how could the truth-value of the proposition be unclear? As an example we turn to three propositions from Peano arithmetic.

- *2 + 3 = 5.*
- *The sum of any two odd numbers is even.*
- *There is an infinity of prime numbers.*

Each of these can be easily proven.[3] And yet there is an interesting difference between the mental picture we have of these theorems. The first is completely clear and uninteresting – it is mentally as well as logically determinate. The second is already somewhat more interesting. Although it is clear enough that adding two odd numbers creates an even one, this theorem allows us a glimpse at the natural numbers as a whole – no matter how large, they are all neatly ordered according to the pattern *even, odd, even, odd*, and addition relates to this pattern in a special way, part of which is captured in the stated theorem. But the last theorem is in many ways the most interesting. Its proof can be understood by any intelligent 12-year old not completely devoid of mathematical insight, but the theorem is more than just its proof. It tells us that no matter how many prime numbers we know, there are still infinitely more we do not. Walking along the line of natural numbers, we encounter quite some primes, but there are always a few more just ahead; and as our gaze approaches the horizon we get the quaint feeling that there is always a prime number just at the edge of our vision, hinting of hidden brothers and sisters even farther away. Yet stepping through the door of this theorem is only the beginning...

3.3 Free play among the determinate

We notice how irregularly the primes are distributed among the numbers, and no matter how hard we try to find a definite pattern according to which they are spread out, it keeps eluding us. They do seems to become less and less frequent as we walk farther, though now and then there are small groups that cluster together; sometimes even a pair of them right next to each other. We start looking for these twins, but cannot make out how many of them populate these regions – they become rare after a while, but do they ever disappear? Retracing our steps, we notice many regularities that escaped us before: diverse families of primes that share a wide variety of characteristic properties, little rules that yield places where a prime can be found for sure – but still no overall pattern becomes clear. We notice apparent regularities, and questions start buzzing in our heads: Are there indefinitely many primes of the form $n^2 + 1$? Can every even number be written as the sum of at most two primes? How many

[3]The first is trivial; the second is clear after a moment's reflection. An easy proof of the infinity of prime numbers can be found at `http://www.hermetic.ch/pns/proof.htm`.

primes are there in any given interval? Wherever we look, new patterns seem to spring up, new questions appear – answers are somewhat rarer, though by no means uncommon, and all the time we are getting a better and better feeling for that strange, complex and incredibly rich family of prime numbers. For every unknown area we explore, two new horizons lure us towards even farther discoveries, and it seems as if an eternity wandering these abstract realms would be far too short to get to know their finer details.

The concept 'prime number' can be defined in a few words, and all the truths concerning them are determined by a few axioms and rules. Yet when we turn from matters of logic to matters of the mind, we discover that these truths are often not directly known; we experience the intricate patterns they form across the wide and varied mathematical 'mindscape'. Problems seem to appear from nowhere, and their solutions only open windows to new thoughts we had not previously dreamt of – the country of the primes is always richer then we can imagine, and it is only a small part of the entire world of Peano arithmetic. No matter how strict and determinate the rules underlying this world may be, to the mind it appears as a wonderful realm full of surprises, mysteries and unknown nooks and crannies which may yield to wide vistas when carefully explored. The determinate character of the mindscape does not force the mind to follow well-trodden, linear paths, but allows it to wander freely, to roam the countryside at will. There is such a thing as a free play among the determinate.

4 Conclusions

The previous section was hardly an example of strict reasoning; but then the possibility of a free play cannot be proven by argument, but only by demonstration. If we are to convince someone that mathematics is beautiful, we cannot possibly confine ourselves to analysing its properties; we have to try and offer a glimpse into the rich aesthetic realm of abstract thought. The previous discussion certainly fails to do this in a way sufficient to convert any sceptics – a much longer and more encompassing treatise would be needed for that.[4] The very least I hope to have shown is that the tension between Kant's concept of the free play of the cognitive powers and the logically determinate realm of mathematical research is largely illusory. Logical determinateness does not imply mental determinateness, and it is the latter, not the former, that would make Kant's free play impossible. Interesting mathematical theories are so complex that the mind may wander through them freely, experiencing a genuine and deep sense of beauty.

[4]Not the worst attempt to do something like this is Rudy Rucker's 1982 book *Infinity and the Mind*, which takes the reader on a tour through the infinite in all its forms. The beauty of the mathematical theory of infinities is exposed at length, in a relatively accessible way.

References

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